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Phase Transitions in the Ashkin–Teller Model

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For some values of the coupling constants, a proof of the existence of two phase transitions in the Ashkin–Teller model is given. Only correlation inequalities are used.

KEY WORDS: Ashkin–Teller model; phase transitions; correlation inequalities.

Some years ago, Wegner published a note[1] on the Ashkin–Teller model[2] in which he pointed out that there are two phase transitions with symmetry breakdown for some values of the coupling constants. I would like to give a proof of the above statement, which is based on Griffiths inequalities [see (A.1) and (A.2) of the Appendix]. In the spin language, the model can be described as follows: for each site x of the lattice $\mathbb{Z}^d$, $d \geqslant 2$, there are two independent spin variables $\sigma(x) = \pm 1$ and $\tau(x) = \pm 1$. The Hamiltonian is

$$H = -\sum_{\langle xy \rangle} \lambda_1 \sigma(x)\sigma(y) + \lambda_1' \tau(x)\tau(y) + \lambda_2 \sigma(x)\sigma(y)\tau(x)\tau(y) \qquad (1)$$

where $\langle xy \rangle$ denotes a pair of nearest neighbors on the lattice. The coupling constants are positive and without loss of generality $\lambda_2 = 1$. The internal symmetry group G contains four elements: the identity I, the transformation I_σ, resp. I_τ, which reverses all spins $\sigma(x)$, resp. $\tau(x)$, and the transformation $I_\sigma \cdot I_\tau$, which reverses all spins. There are three subgroups in G: $G_1 = \{I, I_\sigma \cdot I_\tau\}$, $G_2 = \{I, I_\sigma\}$ and $G_3 = \{I, I_\tau\}$. At high temperature there is a unique equilibrium state, which is of course G invariant. Let $\lambda_1 + \lambda_1' < 1$. In this case, there are two phase transitions with symmetry breakdown at T_1 and at $T_2 < T_1$. At the first transition the symmetry group G is broken, but not completely: there exists an equilibrium state $\langle \cdot \rangle^+$ such that

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$\langle\sigma(x)\tau(x)\rangle^+ > 0$ for all $T < T_1$. The spins $\sigma(x)$ and $\tau(x)$ become positively correlated in the state $\langle\cdot\rangle^+$. However, *all* equilibrium states are G_1 invariant for $T_2 < T < T_1$. In particular, $\langle\sigma(x)\rangle^+ = \langle\tau(x)\rangle^+ = 0$, since the observables $\sigma(x)$ and $\tau(x)$ are not G_1 invariant. At the second phase transition the subgroup G_1 is broken and $\langle\sigma(x)\rangle^+ > 0$, $\langle\tau(x)\rangle^+ > 0$. Let $\lambda_1 > \lambda_1' + 1$. For this choice of the coupling constants there are again two phase transitions at T_3 and at $T_4 < T_3$. At the first transition the group G is partially broken: $\langle\sigma(x)\rangle^+ > 0$ for all $T < T_3$, but *all* equilibrium states are G_3 invariant for $T_4 < T < T_3$. In particular, $\langle\tau(x)\rangle^+ = \langle\sigma(x)\tau(x)\rangle^+ = 0$. At the second phase transition the subgroup G_3 is broken and $\langle\tau(x)\rangle^+ > 0$, $\langle\sigma(x)\tau(x)\rangle^+ > 0$. A similar situation occurs for $\lambda_1' > \lambda_1 + 1$, where the group G is partially broken at the first phase transition, all equilibrium states being G_2 invariant up to the second phase transition.

Using the results of Lebowitz,[3] one can prove, for (almost) all temperatures, that there are two pure phases between $T_2 < T < T_1$, resp. $T_4 < T < T_2$, and four pure phases below T_2 and T_4. We recall that a pure phase is an extremal translation-invariant equilibrium state. Therefore all translation-invariant equilibrium states are convex combinations of two, resp. four, equilibrium states. The state $\langle\cdot\rangle^+$ is always a pure phase. Below T_2 or T_4 the other pure phases are obtained by the action of the group G on $\langle\cdot\rangle^+$ and therefore the pure phases are naturally labeled by the elements of G. In the case $\lambda_1 + \lambda_1' < 1$, the two pure phases between $T_2 < T < T_1$ are labeled by the elements of the quotient group G/G_1, since the actions of I_σ and I_τ on $\langle\cdot\rangle^+$ give the same pure phase. Similarly, for $\lambda_1 > \lambda_1' + 1$, and for $T_4 < T < T_3$, the two pure phases are labeled by the elements of G/G_3.

Remarks. If $\lambda_1 = \lambda_1'$, the internal symmetry group is larger than the group G defined above. Indeed, the system is invariant under the symmetry transformation which exchanges $\sigma(x)$ and $\tau(x)$ for all x. Therefore the symmetry group contains eight elements. However, the pure phases, at low temperature, are still labeled by the subgroup G. This is typical for *ferromagnetic* Abelian spin systems.[4]

Proofs. (a) *Construction of the State* $\langle\cdot\rangle^+$. This is a standard construction. Let Λ be a finite subset of $\mathbb{Z}^d$. Let $\langle\cdot\rangle_\Lambda^+$ be the Gibbs state for the finite volume Λ with $+$ boundary condition. In other words, $\langle\cdot\rangle_\Lambda^+$ is the Gibbs state defined, as usually, by the Hamiltonian given by (1), where the sum is restricted over all $\langle xy\rangle$ such that at least one element belongs to Λ, and if $x \notin \Lambda$, $\sigma(x) = 1$. For any finite subset A of Λ, let $\sigma_A = \prod_{x\in A}\sigma(x)$. By correlation inequality (A.1), $\langle\sigma_A\rangle_{\Lambda_1}^+ \geqslant \langle\sigma_A\rangle_{\Lambda_2}^+ \geqslant 0$ if $\Lambda_1 \subset \Lambda_2$. Therefore $\lim_\Lambda\langle\sigma_A\rangle_\Lambda^+ = \langle\sigma_A\rangle^+$ exists, when Λ tends to $\mathbb{Z}^d$, for any finite A. These limits define an extremal translation-invariant equilibrium state $\langle\cdot\rangle^+$. The state $\langle\cdot\rangle^+$ has two important properties. From (A.1), $\langle\sigma_A\rangle_\Lambda^+ \geqslant \langle\sigma_A\rangle_\Lambda^f$, where $\langle\cdot\rangle_\Lambda^f$ is the Gibbs state defined in Λ with free boundary condition. [In the Hamiltonian (1) the sum is restricted over all $\langle xy\rangle$ inside Λ.] Since $\langle\sigma_A\rangle_{\Lambda_1}^f \leqslant \langle\sigma_A\rangle_{\Lambda_2}^f$ for $\Lambda_1 \subset \Lambda_2$, one has $\langle\sigma(x)\sigma(y)\rangle^+ > 0$ and $\langle\tau(x)\tau(y)\rangle^+ > 0$ for all $\langle xy\rangle$. Using (A.2), one sees that $\langle\sigma_A\rangle^+ > 0$ for all finite A, such that σ_A can be written as a product of $\sigma(x)\sigma(y)$ and $\tau(x)\tau(y)$ for different $\langle xy\rangle$. It is not difficult to see that these observables are exactly those σ_A which are G invariant. Let $\langle\cdot\rangle$ be *any* equilibrium state. If $\langle\sigma_A\rangle^+ = 0$, then $\langle\sigma_A\rangle = 0$. This follows from the fact that all equilibrium states can be expressed as convex combination of equilibrium states obtained by suitable boundary conditions. The above property is true for these states as a consequence of (A.1). In particular, at high temperature, $\langle\sigma_A\rangle = 0$ for any observable σ_A, which is not G invariant, and for any equilibrium state.

(b) $\lambda_1 + \lambda_1' < 1$. Using (A.1) $\langle\sigma(x)\tau(x)\rangle^+(\lambda_1,\lambda_1') \geqslant \langle\sigma(x)\tau(x)\rangle^+(0,0)$. Therefore the Hamiltonian on the right-hand side of this inequality is given by (1) with $\lambda_1 = \lambda_1' = 0$ and $\lambda_2 = 1$. It is not difficult to see that $\langle\sigma(x)\tau(x)\rangle^+(0,0) = \langle\sigma(x)\rangle_I^+(1)$, where $\langle\cdot\rangle_I^+(1)$ denotes the state with + boundary condition of the Ising model

$$H = -\sum_{\langle xy\rangle} \mu\sigma(x)\sigma(y) \tag{2}$$

with coupling constant $\mu = 1$. Let $T_c(\mathrm{Is})$ be the critical temperature of the Ising model with $\mu = 1$. Therefore $\langle\sigma(x)\tau(x)\rangle^+(\lambda_1,\lambda_1') > 0$ for $T < T_c(\mathrm{Is})$. On the other hand, $\langle\sigma(x)\rangle^+(\lambda_1,\lambda_1') \leqslant \langle\sigma(x)\rangle_I^+(\lambda_1+\lambda_1')$. A similar inequality holds for $\langle\tau(x)\rangle^+(\lambda_1,\lambda_1')$. Indeed by adding $-h\sum_x\sigma(x)\tau(x)$ to (1) and letting $h\to\infty$, $\sigma(x) = \tau(x)$ in this limit. Thus $0 \leqslant \langle\sigma(x)\rangle^+(\lambda_1,\lambda_1') + \langle\tau(x)\rangle^+(\lambda_1,\lambda_1') \leqslant 2\langle\sigma(x)\rangle_I^+(\bar\mu)$, with $\bar\mu = \lambda_1 + \lambda_1' < 1$. Therefore $\langle\sigma(x)\rangle^+(\lambda_1,\lambda_1') = \langle\tau(x)\rangle^+(\lambda_1,\lambda_1') = 0$ for all temperatures $T > \bar\mu\cdot T_c(\mathrm{Is})$. Using the properties of $\langle\cdot\rangle^+$ and correlation inequality (A.2), one sees that $\langle\sigma_A\rangle^+ > 0$ if and only if σ_A is G_1 invariant, for all $T_2 < T < T_1$, where $T_1 \geqslant T_c(\mathrm{Is})$ and $T_2 \leqslant \bar\mu\cdot T_c(\mathrm{Is})$. Therefore all states are G_1 invariant for these temperatures.

(c) $\lambda_1 > \lambda_1' + 1$. Clearly (A.1) implies $\langle\sigma(x)\rangle^+(\lambda_1,\lambda_1') \geqslant \langle\sigma(x)\rangle_I^+(\lambda_1) > 0$ if $T \leqslant \lambda_1 T_c(\mathrm{Is})$. Let $\langle\tau(x)\,|\,\sigma\rangle$ be the conditional expectation value of $\tau(x)$, given the values of $\sigma(y)$ for all y. In other words, one keeps the values of $\sigma(y)$ fixed, and sums over all variables $\tau(y)$. By (A.1) this quantity is dominated by its value for $\sigma(y) = 1$ for all y, which is equal to $\langle\tau(x)\rangle_I^+(\lambda_1'+1)$. Since $\langle\tau(x)\rangle^+ = \langle\langle\tau(x)\,|\,\sigma\rangle\rangle^+$, $\langle\tau(x)\rangle^+ \leqslant \langle\tau(x)\rangle_I^+(\lambda_1'+1)$. Therefore $\langle\tau(x)\rangle^+ = 0$ for all $T \geqslant (1+\lambda_1')\cdot T_c(\mathrm{Is})$. Since $\langle\tau(x)\rangle^+ = \langle\sigma(x)\sigma(x)\tau(x)\rangle^+ \geqslant \langle\sigma(x)\tau(x)\rangle^+\langle\sigma(x)\rangle^+$, one also has $\langle\sigma(x)\tau(x)\rangle^+ = 0$ for $T \geqslant (1+\lambda_1')\cdot T_c(\mathrm{Is})$. The rest of the proof is analogous to the former case.

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APPENDIX: GRIFFITHS INEQUALITIES

Let Λ be a finite subset of $\mathbb{Z}^d$, and let $\langle \cdot \rangle(\mathbf{J})$ be the Gibbs state in Λ given by the Hamiltonian

$$H = - \sum_{A \subset \Lambda} J(A)\sigma_A$$

The set of all coupling constants $J(A)$ is denoted by $\mathbf{J}$. The following correlation inequalities hold:

(a) If $|J_1(A)| \leqslant J_2(A)$ for all $A \subset \Lambda$,

$$|\langle \sigma_B \rangle(\mathbf{J}_1)| \leqslant \langle \sigma_B \rangle(\mathbf{J}_2) \tag{A.1}$$

(b) If $J(A) \geqslant 0$ for all $A \subset \Lambda$,

$$\langle \sigma_B \sigma_C \rangle(\mathbf{J}) \geqslant \langle \sigma_B \rangle(\mathbf{J})\langle \sigma_C \rangle(\mathbf{J}) \tag{A.2}$$

For a proof see, e.g., Ref. 5.

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